The Significance of a signal

The case of Pentaquark

The pentaquark is a baryon with five valence quarks.
The clearest signature is that of a

$$u\,u\,d\,d\,\bar{s}\ , \qquad S = +1$$

pentaquark, the unique baryon with positive strangeness.

The $\bar{s}$ antiquark cannot annihilate with the u or d quark by the strong interaction.
Some models predict a mass around 1.5 GeV and a very small width ($\simeq 0.015$ GeV)

The recent pentaquark saga began at 2002 PANIC conference when Nakano measured the following reaction on a Carbon nucleus

$$\gamma n \to \Theta^+ K^- \to K^+ K^- n$$

$$\gamma p \rightarrow K^+ \Lambda(1250) \rightarrow K^+ K^- p$$

$$\gamma n \rightarrow K^- \Theta \rightarrow K^- K^+ n$$

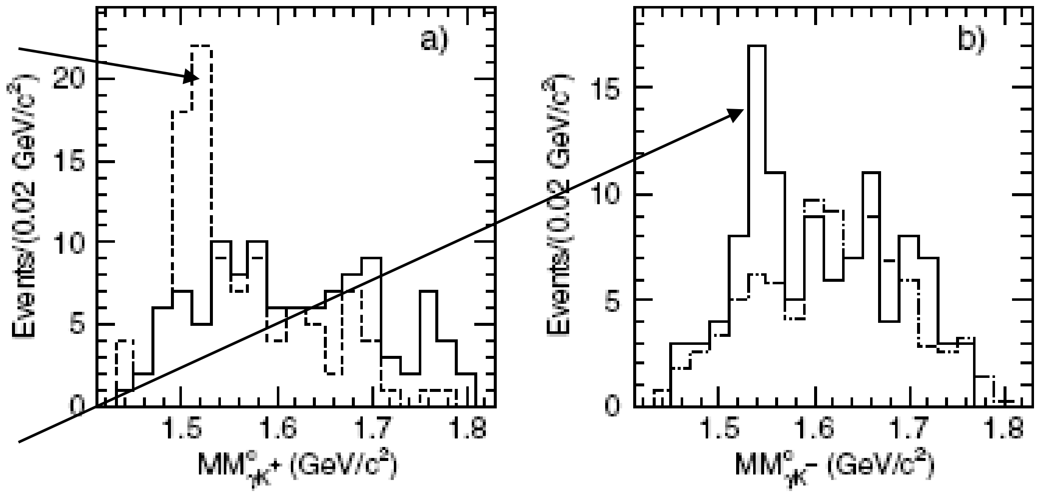


FIG. 3. (a) The $MM^c_{\gamma K^+}$ spectrum [Eq. (2)] for K^+K^- productions for the signal sample (solid histogram) and for events from the SC with a proton hit in the SSD (dashed histogram). (b) The $MM^c_{\gamma K^-}$ spectrum for the signal sample (solid histogram) and for events from the LH_2 (dotted histogram) normalized by a fit in the region above 1.59 GeV$/c^2$.

The first result

PRL 91(2003)012002

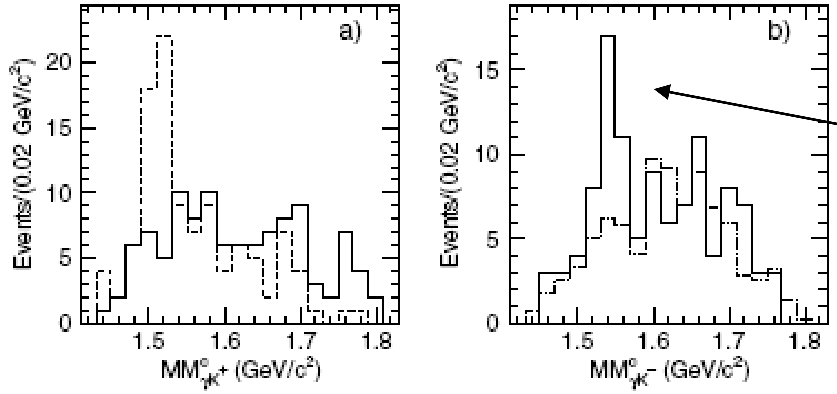
$$\left(\sum_{in} E_{in} - \sum_{fin} E_{fin}\right)^2 - \left(\sum_{in} \vec{p}_{in} - \sum_{fin} \vec{p}_{fin}\right)^2$$

The neutron presence was detected by the $MM_{\gamma K^+K^-}$ missing mass

The $\gamma p \rightarrow K^+K^-p$ reaction was eliminated by direct proton detection.

The neutron was reconstructed from the missing momentum and energy of K^+ and K^-.

The background was measured from a LH_2 target.



4.6 sigma!

Is it convincing???

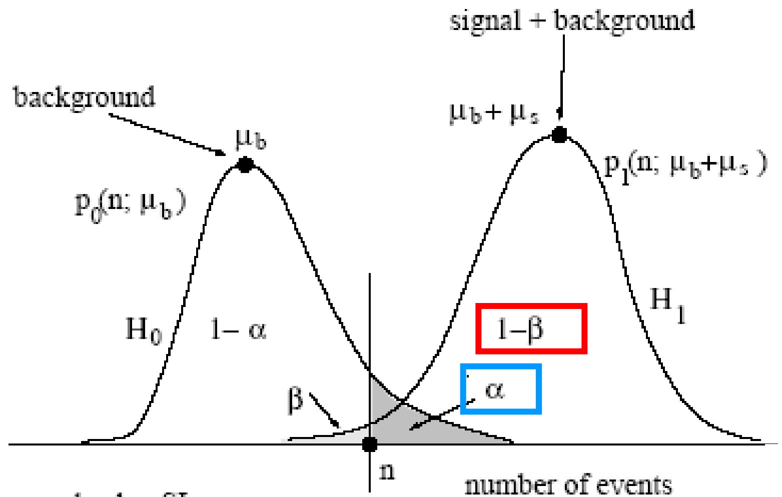
FIG. 3. (a) The $MM^c_{\gamma K^+}$ spectrum [Eq. (2)] for K^+K^- productions for the signal sample (solid histogram) and for events from the SC with a proton hit in the SSD (dashed histogram). (b) The $MM^c_{\gamma K^-}$ spectrum for the signal sample (solid histogram) and for events from the LH_2 (dotted histogram) normalized by a fit in the region above 1.59 GeV/c^2.

012002-3

The background level in the peak region is estimated to be $17.0 \pm 2.2 \pm 1.8$, where the first uncertainty is the error in the fitting in the region above 1.59 GeV/c^2 and the second is a statistical uncertainty in the peak region. The combined uncertainty of the background level is ± 2.8. The estimated number of the events above the background level is 19.0 ± 2.8, which corresponds to a Gaussian significance of $4.6^{+1.2}_{-1.0}\sigma$ $(19.0/\sqrt{17.0} = 4.6)$.

Very important notation:

- Random values before the exp. : M
- Measured values after the experiment: m
- True value: μ



α = backg. SL
β = signal 1− CL
$1-\beta$ =signal CL or power of the test

$\alpha \leq 2.8\cdot 10^{-7}$ 5σ discovery
$\alpha \leq 1.3\cdot 10^{-3}$ 3σ strong evidence
$\alpha \leq 2.3\cdot 10^{-2}$ 2σ weak evidence

$$1-\alpha=\Phi(Z) \quad \rightarrow \quad z=\Phi^{-1}(1-\alpha)$$

where

$$\Phi(Z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{Z}\exp(-t^2/2)\,dt=\frac{1+\mathrm{erf}(Z/\sqrt{2})}{2}$$

true Hypothesis	Decision H_0	Decision H_1
H_0 **background**	correct decision $1-\alpha$ **good rejection**	Type I error α **false acceptance**
H_1 **signal + background**	Type II error β **false exclusion**	Correct decision $1-\beta$ **good acceptance**

Discovery Probability or Discovery Potential (DP): the power $1-\beta$ **when the critical value** n **is decided** ***before*** **the measurement and when** $p(n;\mu_b+\mu_s)$ **is** ***true***.

Poissonian Signal detection

There are many formulas used for detecting a signal over the background (3σ, 5σ, 6σ, and so on)
$N = N_s + N_b$ are the registered counts

$$S_0 = \frac{N - N_b}{\sqrt{N + N_b}} = \frac{N_b + N_s - N_b}{\sqrt{N + N_b}} = \frac{N_s}{\sqrt{N + N_b}}$$

Parameter estimation

$$S_b = \frac{N - \mu_b}{\sqrt{\mu_b}} = \frac{N_b + N_s - \mu_b}{\sqrt{\mu_b}} \simeq \frac{N_s}{\sqrt{\mu_b}}$$

Hypothesis test

This is the most common

$$S_s = \frac{N - \mu_b}{\sqrt{\mu_s}} = \frac{N_b + N_s - \mu_b}{\sqrt{\mu_s}} \simeq \frac{N_s}{\sqrt{\mu_s}}$$

WRONG

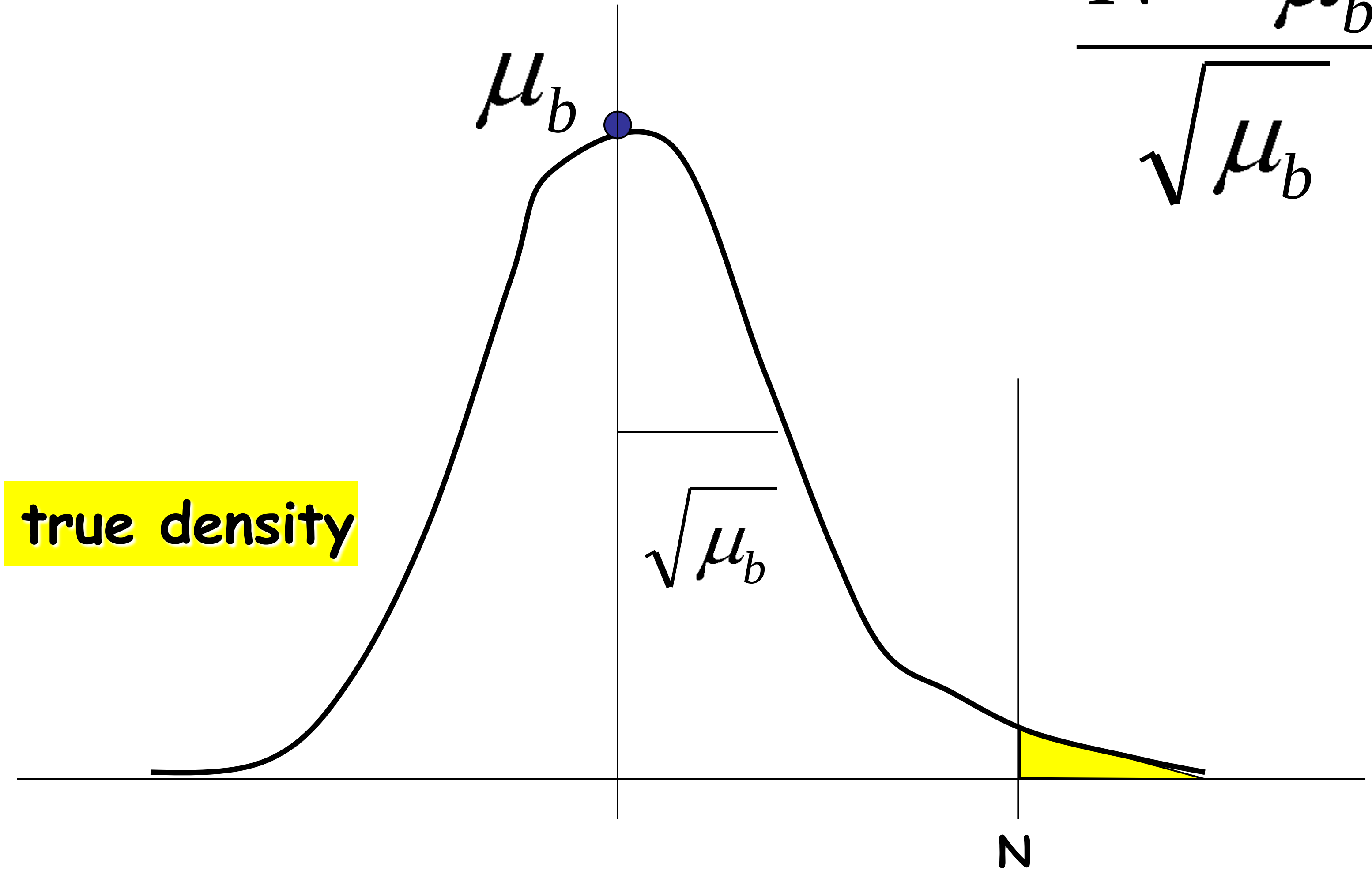
$$S_{sb} = \sqrt{N} - \sqrt{\mu_b} = \sqrt{N_s + N_b} - \sqrt{\mu_b}$$

Recently Proposed (hypothesis test)

Please take care of the notation: often μ is exchanged with N_b and so on, the formulae are obscure and used improperly!!

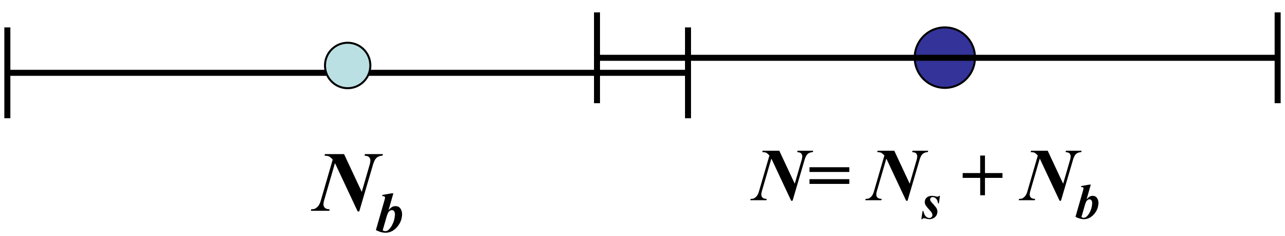
Hypothesis test I

$$\frac{N - \mu_b}{\sqrt{\mu_b}}$$



Parameter estimation

$$N - N_b \pm \sqrt{N + N_b} \cong N_s \pm \sqrt{N_s + 2N_b}$$



$$\frac{N_s}{\sqrt{N_s + 2N_b}}$$

Poissonian Signal detection

When the background is well known people use

$$S_b = \frac{N - \mu_b}{\sqrt{\mu_b}}$$

Recently Bityukov and Krasnikov (2000) proposed

$$S_{sb} = \sqrt{N} - \sqrt{\mu_b} = \sqrt{N_s + N_b} - \sqrt{\mu_b}$$

Proof: In gaussian approx ($\mu_b > 10$), the abscissa n satisfies the equation

$$t = \frac{N - \mu_b}{\sqrt{\mu_b}} = \frac{\mu_s + \mu_b - N}{\sqrt{\mu_s + \mu_b}} \quad \Rightarrow \quad N = \sqrt{\mu_b(\mu_s + \mu_b)}\,, \quad t = \sqrt{\mu_s + \mu_b} - \sqrt{\mu_b}$$

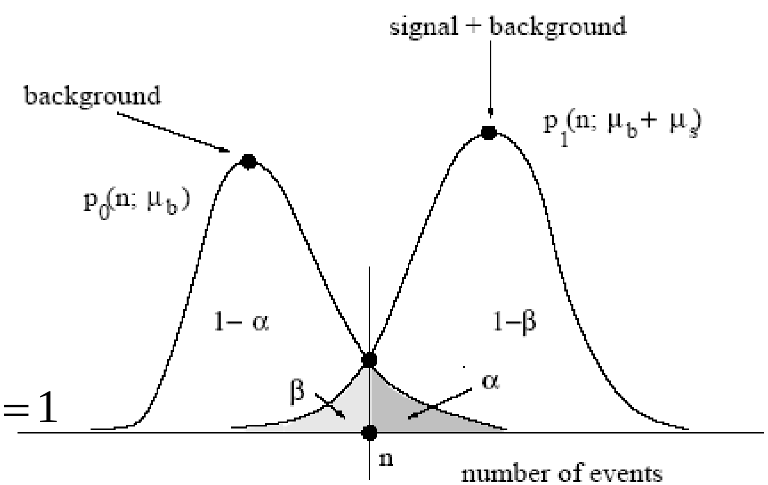
Therefore, one can define the statistic

$$S_{bs} = 2\,(\sqrt{N} - \sqrt{\mu_b})$$

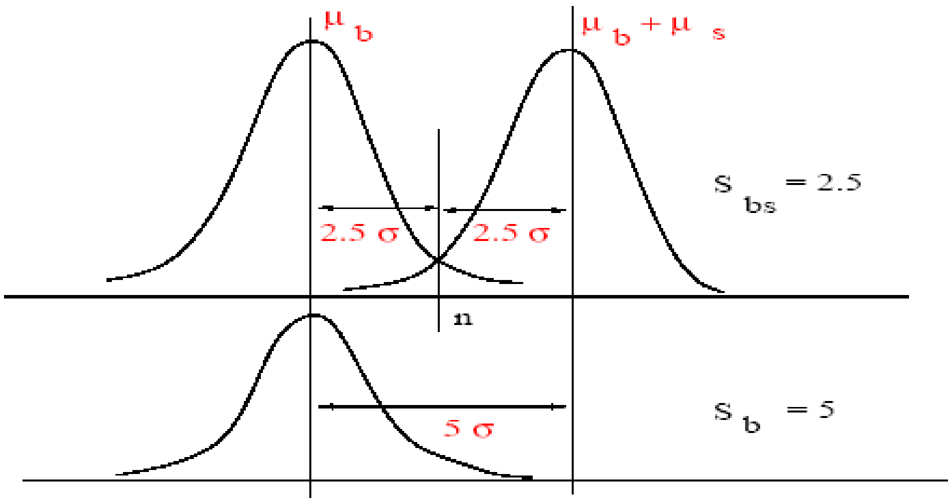
with expectation value

$$\langle S_{bs} \rangle = 2\,(\sqrt{\mu_b + \mu_s} - \sqrt{\mu_b})$$

and unit variance: $\mathrm{Var}[S_{bs}] = 4\ \mathrm{Var}[\sqrt{N}] = 4\left(\frac{1}{2\sqrt{N}}\right)^2 N = 1$



Poissonian Signal detection



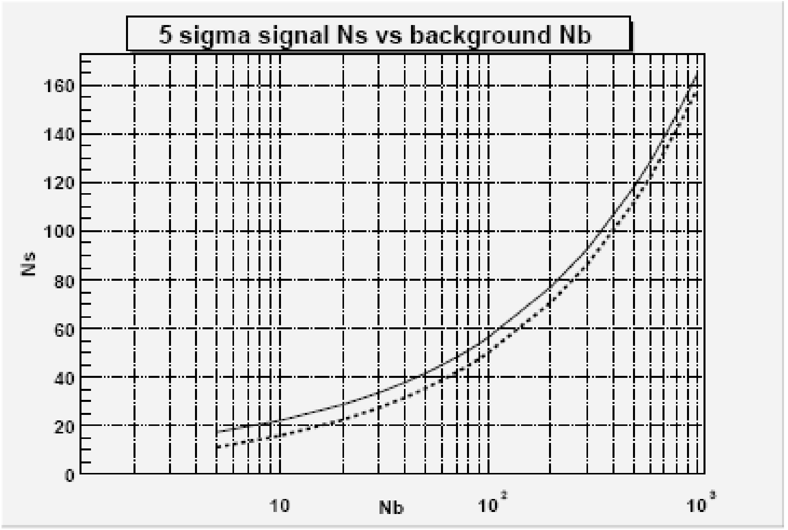
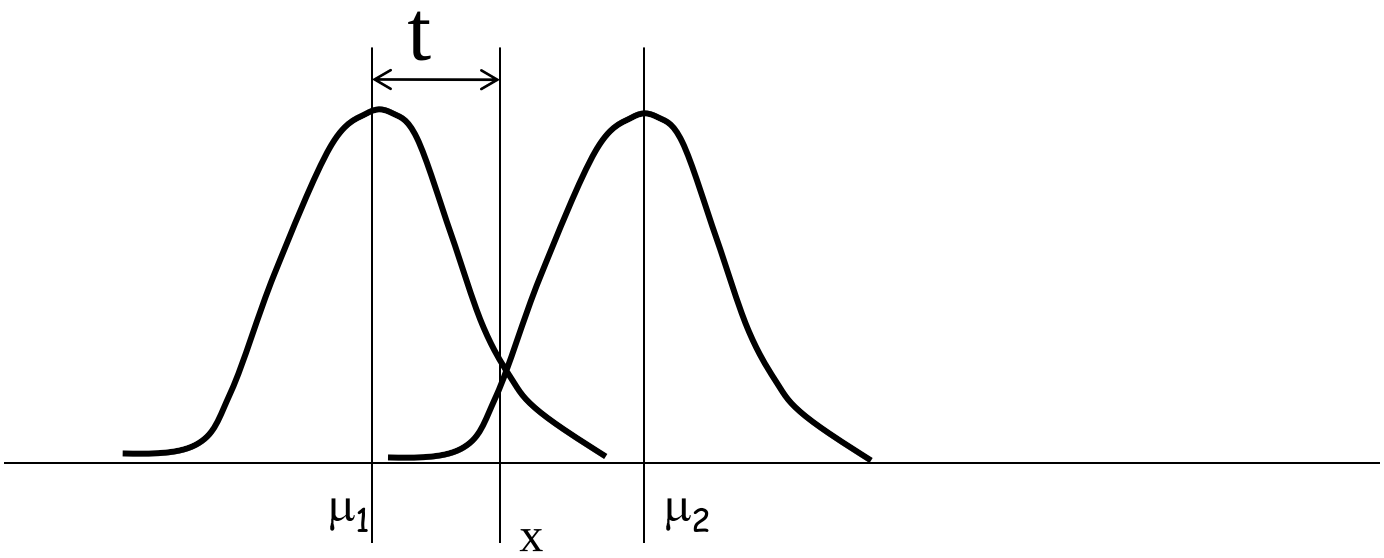


Figure 1: Number N_s of the signal events for $S_b = 5$ (dotted line) and $S_{bs} = 2.5$ (full line) versus the number N_b of background events.

Separation power



$$\frac{x-\mu_1}{\sigma_1}=\frac{\mu_2-x}{\sigma_2}=t \quad \rightarrow \quad x=\frac{\sigma_1\mu_2+\sigma_2\mu_1}{\sigma_1+\sigma_2}$$

Separation power 2t

$$2t=2\,\frac{\mu_2-x}{\sigma_2}=2\,\frac{\mu_2}{\sigma_2}-\frac{2}{\sigma_2}\left[\frac{\sigma_1\mu_2+\sigma_2\mu_1}{\sigma_1+\sigma_2}\right]=2\left[\frac{\mu_2-\mu_1}{\sigma_1+\sigma_2}\right]=\left[\frac{\mu_2-\mu_1}{\sigma_1/2+\sigma_2/2}\right]$$

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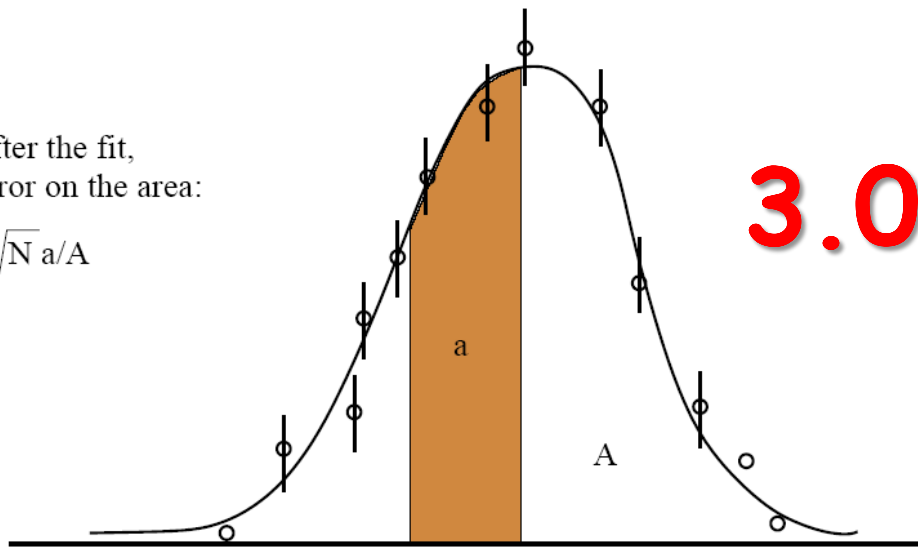
after the fit,
error on the area:

$\sqrt{N}\, a/A$



3.07

$$\frac{19}{\sqrt{19+17+17}} = 2.6$$

$$\frac{19}{\sqrt{17+2.8^2}} = 3.8$$

Observation of an Exotic Baryon with $S = +1$ in Photoproduction from the Proton

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C. Salgado,[26] P. Ambrozewicz,[13] E. Anciant,[5] M. Anghinolfi,[18] B. Asavapibhop,[24] G. Audit,[5] T. Auger,[5] H. Avakian,[3]
H. Bagdasaryan,[28] J. P. Ball,[4] S. Barrow,[14] K. Beard,[21] M. Bektasoglu,[27] M. Bellis,[1] N. Benmouna,[15] B. L. Berman,[15]
C. S. Whisnant,[32] E. Wolin,[3] M. H. Wood,[32] A. Yegneswaran,[3] and J. Yun[28]

(CLAS Collaboration)

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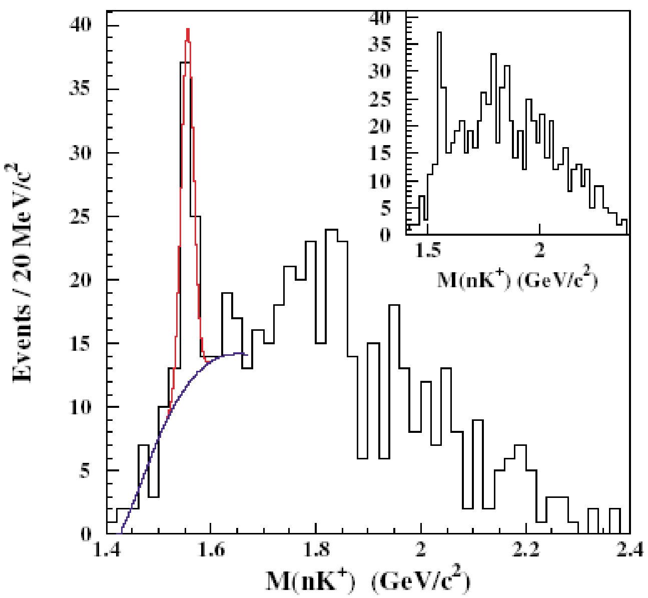

FIG. 4 (color online). The nK^+ invariant mass spectrum in the reaction $\gamma p \rightarrow \pi^+ K^- K^+ (n)$ with the cut $\cos\theta^*_{\pi^+} > 0.8$ and $\cos\theta^*_{K^+} < 0.6$. $\theta^*_{\pi^+}$ and $\theta^*_{K^+}$ are the angles between the π^+ and K^+ mesons and photon beam in the center-of-mass system. The background function we used in the fit was obtained from the simulation. The inset shows the nK^+ invariant mass spectrum with only the $\cos\theta^*_{\pi^+} > 0.8$ cut.

The final nK^+ effective mass distribution (Fig. 4) was fitted by the sum of a Gaussian function and a background function obtained from the simulation. The fit parameters are $N_{\Theta^+} = 41 \pm 10$, $M = 1555 \pm 1$ MeV$/c^2$, and $\Gamma = 26 \pm 7$ MeV$/c^2$ (FWHM), where the errors are statistical. The systematic mass scale uncertainty is estimated to be ± 10 MeV$/c^2$. This uncertainty is larger than our previously reported uncertainty [6] because of the different energy range and running conditions and is mainly due to the momentum calibration of the CLAS detector and the photon beam energy calibration. The statistical significance for the fit in Fig. 4 over a 40 MeV$/c^2$ mass window is calculated as $N_P/\sqrt{N_B}$, where N_B is the number of counts in the background fit under the peak and N_P is the number of counts in the peak. We estimate the significance to be $7.8 \pm 1.0\sigma$. The uncertainty of 1.0σ is due to

Evidence for a narrow |S| = 1 baryon state at a mass of 1528 MeV in quasi-real photoproduction

A. Airapetian,[31] N. Akopov,[31] Z. Akopov,[31] M. Amarian,[8,31] V.V. Ammosov,[23] A. Andrus,[16] E.C. Aschenauer,[8] W. Augustyniak,[30] R. Avakian,[31] A. Avetissian,[31] E. Avetissian,[12] P. Bailey,[16] D. Balin,[22] V. Baturin,[22] M. Beckmann,[7] S. Belostotski,[22] S. Bernreuther,[10] N. Bianchi,[12] H.P. Blok,[21,29] H. Böttcher,[8] A. Borissov,[18]
G. Ybeles Smit,[29] Y. Ye,[5] Z. Ye,[5] S. Yen,[27] W. Yu,[4] B. Zihlmann,[21] H. Zohrabian,[31] and P. Zupranski[30]
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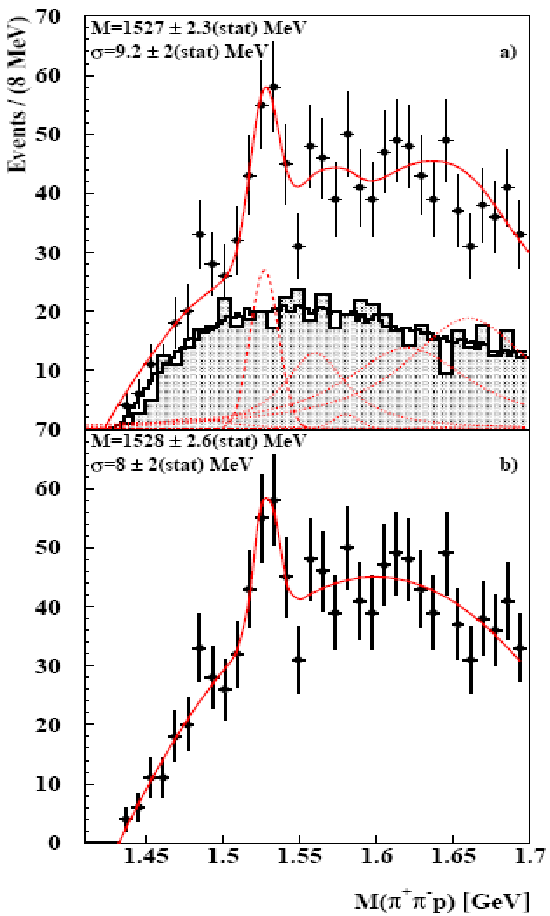
[1] Department of Physics, University of Alberta, Edmonton, Alberta T6G 2J1, Canada



Photoproduction on a deuterium target

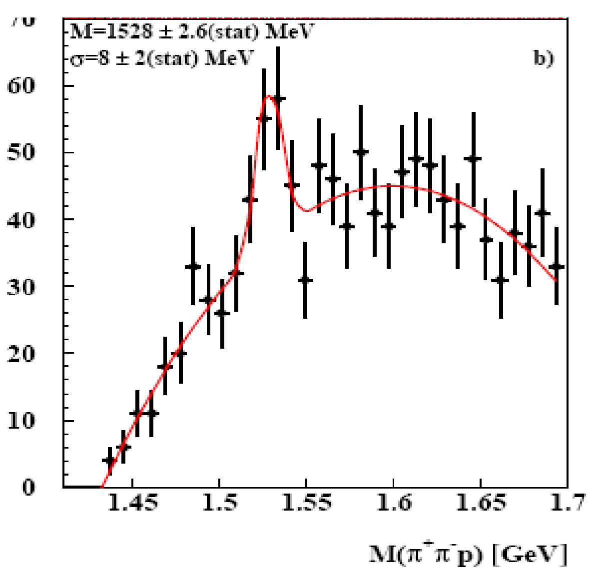
$$\Theta^+ \rightarrow pK^0_S \rightarrow p\pi^+\pi^-$$

FIG. 2: Distribution in invariant mass of the $p\pi^+\pi^-$ system subject to various constraints described in the text. The experimental data are represented by the filled circles with statistical error bars, while the fitted smooth curves result in the indicated position and σ width of the peak of interest. In panel a), the PYTHIA6 Monte Carlo simulation is represented by the gray shaded histogram, the mixed-event model normalised to the PYTHIA6 simulation is represented by the fine-binned histogram, and the fitted curve is described in the text. In panel b), a fit to the data of a Gaussian plus a third-order polynomial is shown.

HERMES : 27.6 GeV positron beam on deuterium

TABLE I: Mass values and experimental widths, with their statistical and systematic uncertainties, for the Θ^+ from the two fits, labelled by a) and b), shown in the corresponding panels of Fig. 2. Rows a') and b') are based on the same background models as rows a) and b) respectively, but a different mass reconstruction expression that is expected to result in better resolution. Also shown are the number of events in the peak N_s and the background N_b, both evaluated from the functions fitted to the mass distribution, and the results for the naïve significance $N_s^{2\sigma}/\sqrt{N_b^{2\sigma}}$ and realistic significance $N_s/\delta N_s$. The systematic uncertainties are common (correlated) between rows of the table.

	Θ^+ mass [MeV]	FWHM [MeV]	$N_s^{2\sigma}$ in $\pm 2\sigma$	$N_b^{2\sigma}$ in $\pm 2\sigma$	naïve signif.	Total $N_s \pm \delta N_s$	signif.
a)	$1527.0 \pm 2.3 \pm 2.1$	$22 \pm 5 \pm 2$	74	145	6.1σ	78 ± 18	4.3σ
a')	$1527.0 \pm 2.5 \pm 2.1$	$24 \pm 5 \pm 2$	79	158	6.3σ	83 ± 20	4.2σ
b)	$1528.0 \pm 2.6 \pm 2.1$	$19 \pm 5 \pm 2$	56	144	4.7σ	59 ± 16	3.7σ
b')	$1527.8 \pm 3.0 \pm 2.1$	$20 \pm 5 \pm 2$	52	155	4.2σ	54 ± 16	3.4σ

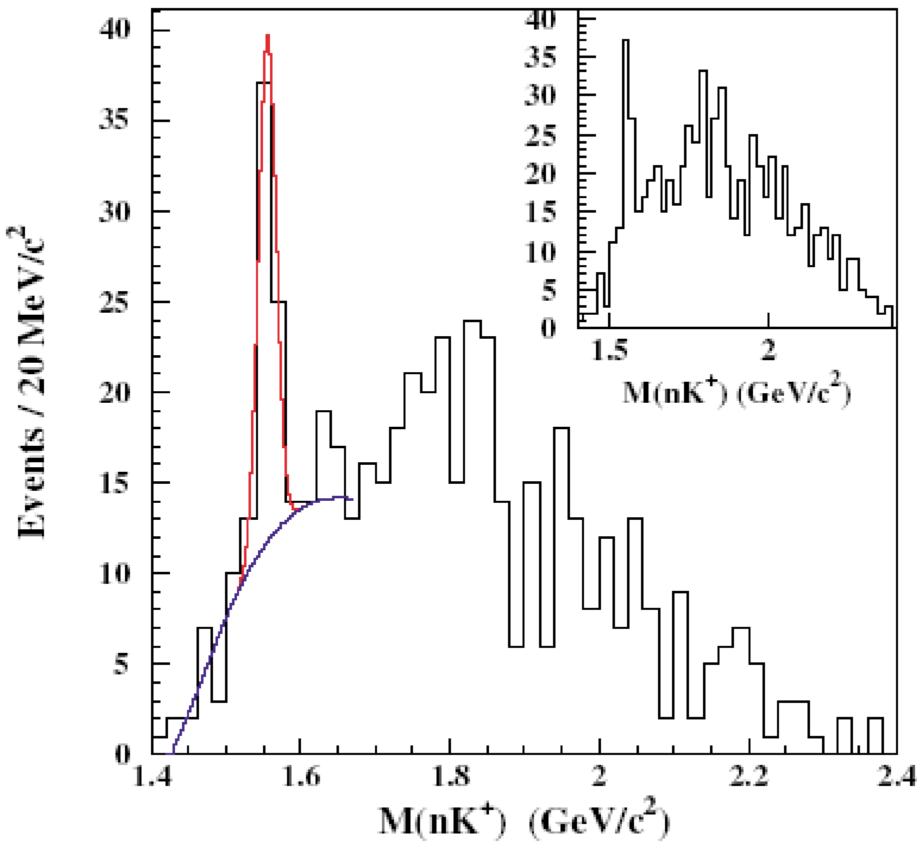


$$S_b = \frac{N - \mu_b}{\sqrt{\mu_b}} = \frac{N_b + N_s - \mu_b}{\sqrt{\mu_b}}$$

$$S_0 = \frac{N - N_b}{\sqrt{N + N_b}} = \frac{N_s + N_b - N_b}{\sqrt{N + N_b}} = \frac{N_s}{\sqrt{N + N_b}}$$

Z=3.84

$74/\sqrt{74+145+74} = 4.3$



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$$\frac{41}{\sqrt{27}} = 7.8$$???

$$\frac{41}{\sqrt{41 + 27 + 27}} = 4.2$$

Statistics

$$\xi_1 = \frac{s}{\sqrt{b}}$$

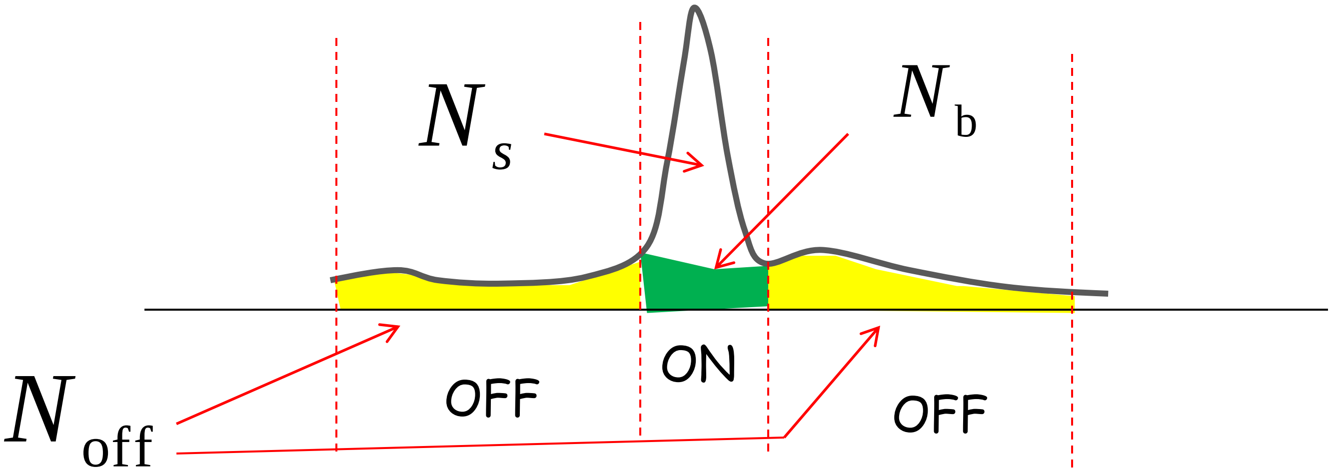
$$\xi_2 = \frac{s}{\sqrt{s+b}}$$

$$\xi_3 = \frac{s}{\sqrt{s+2b}}$$

Experiment	Signal	Background	Publ.	Significance ξ_1	ξ_2	ξ_3
Spring8	19	17	4.6σ	4.6	3.2	2.6
Spring8	56	162		4.4	3.8	2.9
SPAHIR	55	56	4.8σ	7.3	5.2	4.3
CLAS (d)	43	54	5.2σ	5.9	4.4	3.5
CLAS (p)	41	35	7.8σ	6.9	4.7	3.9
DIANA	29	44	4.4σ	4.4	3.4	2.7
ν	18	9	6.7σ	6.0	3.5	3.0
HERMES	51	150	4.3-6.2σ	4.2	3.6	2.7
COSY	57	95	4-6σ	5.9	4.7	3.7
ZEUS	230	1080	4.6σ	7.0	6.4	4.7
SVD	35	93	5.6σ	3.6	3.1	2.4
NOMAD	33	59	4.3σ	4.3	3.4	2.7
NA49	38	43	4.2σ	5.8	4.2	3.4
NA49	69	75	5.8σ	8.0	5.8	4.7
H1	50.6	51.7	5-6σ	7.0	5.0	4.1

No 5σ effect!!

All these methods estimate true values through measured quantities… but …



$$N_{off} \approx \text{Pois}(\lambda\, \mu_b) \quad \text{with } \lambda \text{ known}$$

Consider

$$N_{on} = N_s + N_b$$

$$N_b \cong \frac{N_{off}}{\lambda} \qquad \lambda = \frac{N_{off}}{N_b} \qquad \sigma_b = \frac{\sqrt{N_{off}}}{\lambda} \qquad \lambda = \frac{N_b}{\sigma_b^2}$$

A first rigorous solution

R. Cousins et al, NIM A 595(2008)480

The joint probability of observing $n_{\rm on}$ and $n_{\rm off}$ is the product of Poisson probabilities for $n_{\rm on}$ and $n_{\rm off}$, and can be rewritten as the product of a single Poisson probability with mean $\mu_{\rm tot} = \mu_{\rm on} + \mu_{\rm off}$ for the total number of events $n_{\rm tot}$, and the binomial probability that this total is divided as observed if the binomial parameter ρ is $\rho = \mu_{\rm on}/\mu_{\rm tot} = 1/(1+\lambda)$:

$$\lambda = \mu_{\rm off}/\mu_{\rm on} \xrightarrow{H_0} \mu_{\rm off}/\mu_{\rm b}$$

$$P(n_{\rm on}, n_{\rm off}) = \frac{e^{-\mu_{\rm on}}\mu_{\rm on}^{n_{\rm on}}}{n_{\rm on}!} \times \frac{e^{-\mu_{\rm off}}\mu_{\rm off}^{n_{\rm off}}}{n_{\rm off}!}$$

$$= \frac{e^{-(\mu_{\rm on}+\mu_{\rm off})}(\mu_{\rm on}+\mu_{\rm off})^{n_{\rm tot}}}{n_{\rm tot}!} \qquad (9)$$

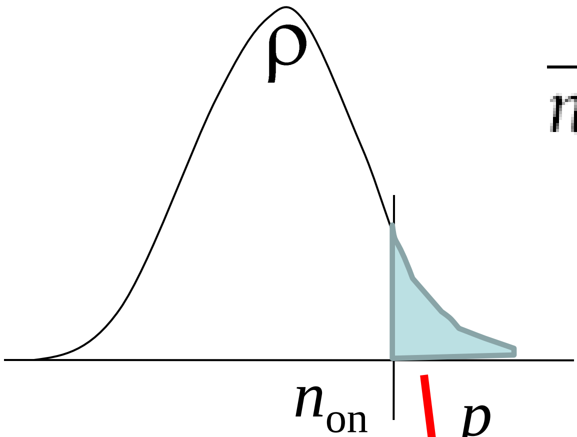
$$\times \frac{n_{\rm tot}!}{n_{\rm on}!(n_{\rm tot}-n_{\rm on})!}\rho^{n_{\rm on}}(1-\rho)^{(n_{\rm tot}-n_{\rm on})} \qquad (10)$$

ρ is known!!!

λ is the known normalization constant supposing that th

on measurement does not contain the signal **(H_0 hyp.)**

A rigorous solution



$$\frac{n_{\rm tot}!}{n_{\rm on}!(n_{\rm tot}-n_{\rm on})!}\rho^{n_{\rm on}}(1-\rho)^{(n_{\rm tot}-n_{\rm on})}$$

$$\rho = \mu_{\rm on}/\mu_{\rm tot} = 1/(1+\lambda)$$

$$\lambda = \mu_{\rm off}/\mu_{\rm on}$$

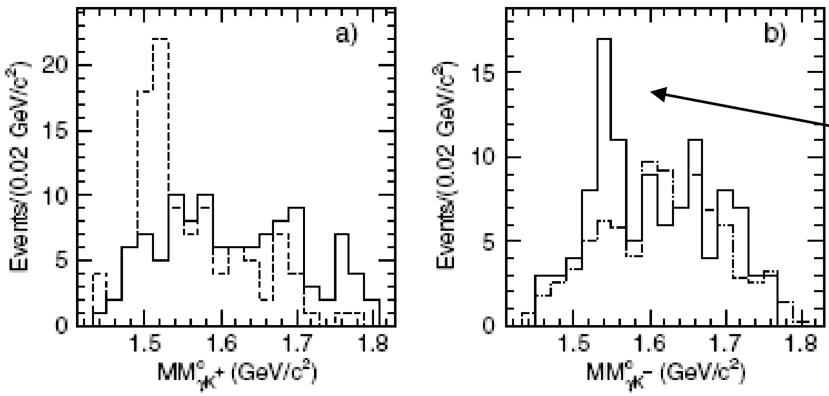
$$p_{\rm Bi} = \sum_{j=n_{\rm on}}^{n_{\rm tot}} P(j|n_{\rm tot};\rho) = B(\rho, n_{\rm on}, 1+n_{\rm off})/B(n_{\rm on}, 1+n_{\rm off})$$

$$Z = \Phi^{-1}(1-p) = -\Phi^{-1}(p)$$

$$Z = \sqrt{2}\,\mathrm{erf}^{-1}(1-2p)$$

where

$$\Phi(Z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{Z}\exp(-t^2/2)\,\mathrm{d}t = \frac{1+\mathrm{erf}(Z/\sqrt{2})}{2} = 1-p$$



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012002-3

4.6 sigma!

Is it convincing???

The background level in the peak region is estimated to be $17.0 \pm 2.2 \pm 1.8$, where the first uncertainty is the error in the fitting in the region above 1.59 GeV/c^2 and the second is a statistical uncertainty in the peak region. The combined uncertainty of the background level is ± 2.8. The estimated number of the events above the background level is 19.0 ± 2.8, which corresponds to a Gaussian significance of $4.6^{+1.2}_{-1.0}\sigma$ $(19.0/\sqrt{17.0} = 4.6)$.

$$p_{\mathrm{Bi}} = \sum_{j=n_{\mathrm{on}}}^{n_{\mathrm{tot}}} P(j|n_{\mathrm{tot}};\rho) \quad = B(\rho, n_{\mathrm{on}}, 1+n_{\mathrm{off}})/B(n_{\mathrm{on}}, 1+n_{\mathrm{off}})$$

$$Z = \sqrt{2}\,\mathrm{erf}^{-1}(1-2p)$$

For the simple on/off problem with $n_{\mathrm{on}} = 140$, $n_{\mathrm{off}} = 100$, and $\tau = 1.2$, the ROOT commands are:

```
double n_on = 140.
double n_off = 100.
double tau = 1.2
double P_Bi = TMath::BetaIncomplete(1./(1.+tau),n_on,n_off+1)
double Z_Bi = sqrt(2)*TMath::ErfInverse(1 - 2*P_Bi)
```

Pentaquark: n_on=36, n_off= 17*2.17 = 36.7,

$\tau = \lambda = 17/2.8^2 = 2.17$, Z=3.07

A 2nd "rigorous" solution

$$\lambda = \frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{on}}} \rightarrow \text{no signal} \rightarrow \tau = \frac{\mu_{\mathrm{off}}}{\mu_{\mathrm{b}}} \qquad \frac{1}{1+\tau} = \frac{\mu_{\mathrm{b}}}{\mu_{\mathrm{b}} + \mu_{\mathrm{off}}}$$

$$\Lambda = \frac{L(\mu)}{L(\text{best})} = \frac{\mu_b^{N_{\mathrm{on}}} \; e^{-\mu_{\mathrm{b}}} \; \mu_{\mathrm{off}}^{N_{\mathrm{off}}} \; e^{-\mu_{\mathrm{off}}}}{N_{\mathrm{on}}^{N_{\mathrm{on}}} \; e^{-N_{\mathrm{on}}} \; N_{\mathrm{off}}^{N_{\mathrm{off}}} \; e^{-N_{\mathrm{off}}}}$$

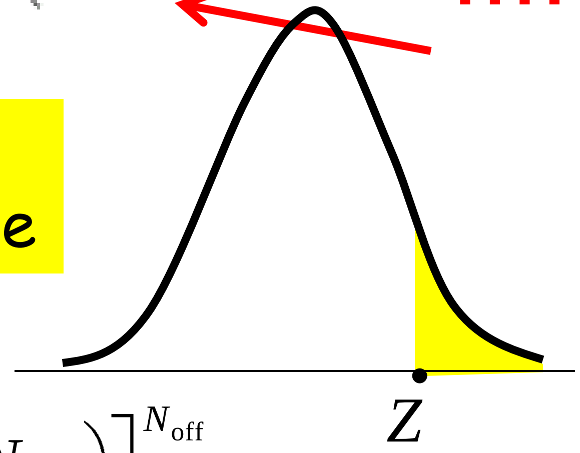
$$\mu_b = \frac{1}{1+\tau}(N_{\mathrm{on}} + N_{\mathrm{off}}) \qquad \mu_{\mathrm{off}} = \frac{\tau}{1+\tau}(N_{\mathrm{on}} + N_{\mathrm{off}})$$

$$\Lambda = \left[\frac{1}{1+\tau}\left(\frac{N_{\mathrm{on}} + N_{\mathrm{off}}}{N_{\mathrm{on}}}\right)\right]^{N_{\mathrm{on}}} \left[\frac{\tau}{1+\tau}\left(\frac{N_{\mathrm{on}} + N_{\mathrm{on}}}{N_{\mathrm{off}}}\right)\right]^{N_{\mathrm{off}}}$$

A 2nd "rigorous" solution

T.-P. Li, Y.-Q. Ma, Astrophys. J. 272 (1983) 317.

!!!!



Approximated Gaussian significance

$$\Lambda = \left[\frac{1}{1+\tau}\left(\frac{N_{\text{on}}+N_{\text{off}}}{N_{\text{on}}}\right)\right]^{N_{\text{on}}} \left[\frac{\tau}{1+\tau}\left(\frac{N_{\text{on}}+N_{\text{on}}}{N_{\text{off}}}\right)\right]^{N_{\text{off}}}$$

$$Z = \sqrt{\chi^2(1)} = \sqrt{-2\ln\Lambda} =$$

$$\sqrt{2}\left[N_{\text{on}}\ln\left(\frac{(1+\tau)N_{\text{on}}}{N_{\text{on}}+N_{\text{off}}}\right) + N_{\text{off}}\ln\frac{1+\tau}{\tau}\left(\frac{N_{\text{off}}}{N_{\text{on}}+N_{\text{off}}}\right)\right]^{1/2}$$

A 2nd rigorous solution

R. Cousins et al, NIM A 595(2008)480

$$\mathscr{L}_{\mathrm{P}} = \frac{(\mu_{\mathrm{s}} + \mu_{\mathrm{b}})^{n_{\mathrm{on}}}}{n_{\mathrm{on}}!} \mathrm{e}^{-(\mu_{\mathrm{s}}+\mu_{\mathrm{b}})} \frac{(\tau\mu_{\mathrm{b}})^{n_{\mathrm{off}}}}{n_{\mathrm{off}}!} \mathrm{e}^{-\tau\mu_{\mathrm{b}}} \tag{20}$$

while for the Gaussian-mean background problem with either absolute or relative σ_{b}, it is

$$\mathscr{L}_{\mathrm{G}} = \frac{(\mu_{\mathrm{s}} + \mu_{\mathrm{b}})^{n_{\mathrm{on}}}}{n_{\mathrm{on}}!} \mathrm{e}^{-(\mu_{\mathrm{s}}+\mu_{\mathrm{b}})} \frac{1}{\sqrt{2\pi\sigma_{\mathrm{b}}^2}} \exp\left(-\frac{(\hat{\mu}_{\mathrm{b}} - \mu_{\mathrm{b}})^2}{2\sigma_{\mathrm{b}}^2}\right) \tag{21}$$

where as discussed below we have explored the effect of truncating the Gaussian pdf in $\hat{\mu}_{\mathrm{b}}$ and renormalizing prior to forming $\mathscr{L}_{\mathrm{G}}$.

Using either $\mathscr{L}_{\mathrm{P}}$ or $\mathscr{L}_{\mathrm{G}}$, one obtains the log-likelihood ratio

$$\Lambda(\mu_{\mathrm{s}}) = \frac{\mathscr{L}(\mu_{\mathrm{s}}, \tilde{\tilde{\mu}}_{\mathrm{b}}(\mu_{\mathrm{s}}))}{\mathscr{L}(\tilde{\mu}_{\mathrm{s}}, \tilde{\mu}_{\mathrm{b}})} \qquad -2\ln\Lambda(\mu_{\mathrm{s}}) < F^{-1}_{\chi^2_1}(1-2\alpha) \tag{22}$$

A 2nd rigourous solution

$$Z_{PL} = \sqrt{-2 \ln \Lambda(\mu_s = 0)} \quad (24)$$

where the likelihood ratio is computed using $\mathscr{L}_P$ or $\mathscr{L}_G$, as appropriate for the problem.

For the on/off problem and $\mathscr{L}_P$, the explicit result obtained from Eq. (24) was given by Li and Ma (their Eq. 17) [8]:

$$Z_{PL} = \sqrt{2}\left(n_{on} \ln \frac{n_{on}(1+\tau)}{n_{tot}} + n_{off} \ln \frac{n_{off}(1+\tau)}{n_{tot}\tau}\right)^{1/2}. \quad (25)$$

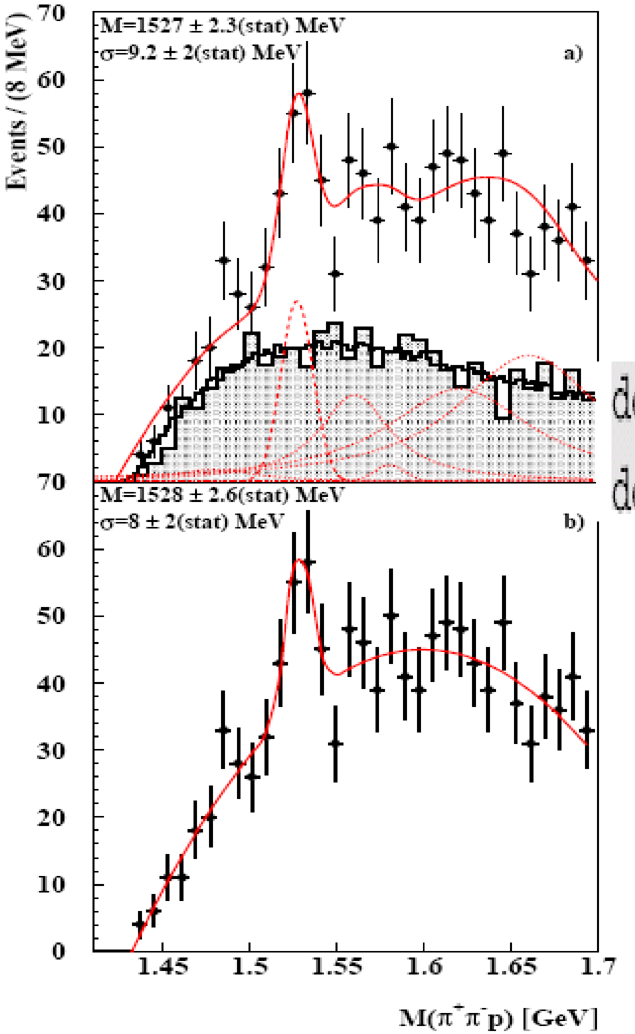
Pentaquark: n_on=36, n_off= 17*2.17 = 36.7,

$\tau = \lambda = 17/2.8^2 = 2.17$, Z=3.25

HERMES : 27.6 GeV positron beam on deuterium

$$S_b = \frac{N - \mu_b}{\sqrt{\mu_b}} = \frac{N_b + N_s - \mu_b}{\sqrt{\mu_b}} \simeq \frac{N_s}{\sqrt{\mu_b}} = 6.1$$

$$74/\sqrt{74+145+74} = 4.3$$



$$S_0 = \frac{N - N_b}{\sqrt{N + N_b}} = \frac{N_b + N_s - N_b}{\sqrt{N + N_b}} = \frac{N_s}{\sqrt{N + N_b}} = 4.3$$

$N_{off} = 145 \qquad N_{on} = 145 + 74 = 219$

$N_{tot} = 145 + 219 = 364 \quad N_b = 145 \quad \lambda = \tau = 1$

```
double P_Bi = TMath::BetaIncomplete(1./(1.+tau),n_on,n_off+1)
double Z_Bi = sqrt(2)*TMath::ErfInverse(1 - 2*P_Bi)
```

Z=3.84 Binomial

$$Z = \sqrt{2}\left[N_{on}\ln\left(\frac{(1+\tau)N_{on}}{N_{on}+N_{off}}\right) + N_{off}\ln\frac{1+\tau}{\tau}\left(\frac{N_{off}}{N_{on}+N_{off}}\right)\right]^{1/2}$$

Z=3.90 Likelihood

Table 1
Test cases and significance results

Reference	[40]	[41]	[42]	[43]	[44]	[44]	[45]	[46]	[47]	[48]
n_{on}	**4**	**6**	**9**	**17**	**50**	**67**	**200**	**523**	**498 426**	**2 119 449**
n_{off}	**5**	18.78	17.83	40.11	**55**	**15**	**10**	**2327**	**493 434**	23 650 096
τ	**5.0**	14.44	4.69	10.56	**2.0**	**0.5**	**0.1**	**5.99**	**1.0**	**11.21**
$\hat{\mu}_{\text{b}}$	1.0	**1.3**	**3.8**	**3.8**	27.5	30.0	100.0	388.6	493 434	**2 109 732**
$s = n_{\text{on}} - \hat{\mu}_{\text{b}}$	3.0	4.7	5.2	13.2	22.5	37	100	**134**	**4992**	**9717**
σ_{b}	0.447	**0.3**	**0.9**	**0.6**	3.71	7.75	31.6	8.1	702.4	433.8
$f = \sigma_{\text{b}}/\hat{\mu}_{\text{b}}$	0.447	0.231	0.237	0.158	0.135	0.258	0.316	0.0207	0.00142	0.000206
Reported p		**0.003**	**0.027**	**2E−06**						
Reported Z		2.7	1.9	**4.6**				**5.9**	**5.0**	**6.4**
See conclusion										
$Z_{\text{Bi}} = Z_{\Gamma}$ binomial	**1.66**	**2.63**	**1.82**	**4.46**	**2.93**	**2.89**	**2.20**	**5.93**	**5.01**	**6.40**
Z_{N} Bayes Gaussian	1.88	2.71	1.94	4.55	3.08	*3.44*	*2.90*	**5.93**	**5.02**	**6.40**
Z_{PL} profile likelihood	1.95	2.81	1.99	4.57	3.02	3.04	2.38	**5.93**	**5.01**	**6.41**
Z_{ZR} variance stabilization	1.93	2.66	1.98	4.22	3.00	3.07	2.39	5.86	**5.01**	**6.40**
Not recommended										
$Z_{\text{BiN}} = s/\sqrt{n_{\text{tot}}/\tau}$	*2.24*	*3.59*	2.17	*5.67*	3.11	**2.89**	**2.18**	6.16	**5.01**	**6.41**
$Z_{\text{nn}} = s/\sqrt{n_{\text{on}} + n_{\text{off}}/\tau^2}$	1.46	*1.90*	1.66	*3.17*	2.82	3.28	*2.89*	*5.54*	**5.01**	**6.40**
$Z_{\text{ssb}} = s/\sqrt{\hat{\mu}_{\text{b}} + s}$	1.50	*1.92*	1.73	*3.20*	3.18	*4.52*	*7.07*	5.88	*7.07*	6.67
$Z_{\text{bo}} = s/\sqrt{n_{\text{off}}(1+\tau)/\tau^2}$	*2.74*	*3.99*	*2.42*	*6.47*	*3.50*	*3.90*	*3.02*	6.31	**5.03**	**6.41**
Ignore σ_{b}										
Z_{P} Poisson: ignore σ_{b}	2.08	2.84	2.14	4.87	*3.80*	*5.76*	*8.76*	*6.44*	*7.09*	6.69
$Z_{\text{sb}} = s/\sqrt{\hat{\mu}_{\text{b}}}$	*3.00*	*4.12*	*2.67*	*6.77*	*4.29*	*6.76*	*10.00*	*6.82*	*7.11*	6.69
Unsuccessful ad hockery										
Poisson: $\mu_{\text{b}} \to \hat{\mu}_{\text{b}} + \sigma_{\text{b}}$	1.56	2.51	1.64	**4.47**	3.04	*4.24*	*5.51*	6.01	*6.09*	**6.39**
$s/\sqrt{\hat{\mu}_{\text{b}} + \sigma_{\text{b}}}$	*2.49*	*3.72*	*2.40*	*6.29*	*4.03*	*6.02*	*8.72*	*6.75*	*7.10*	6.69

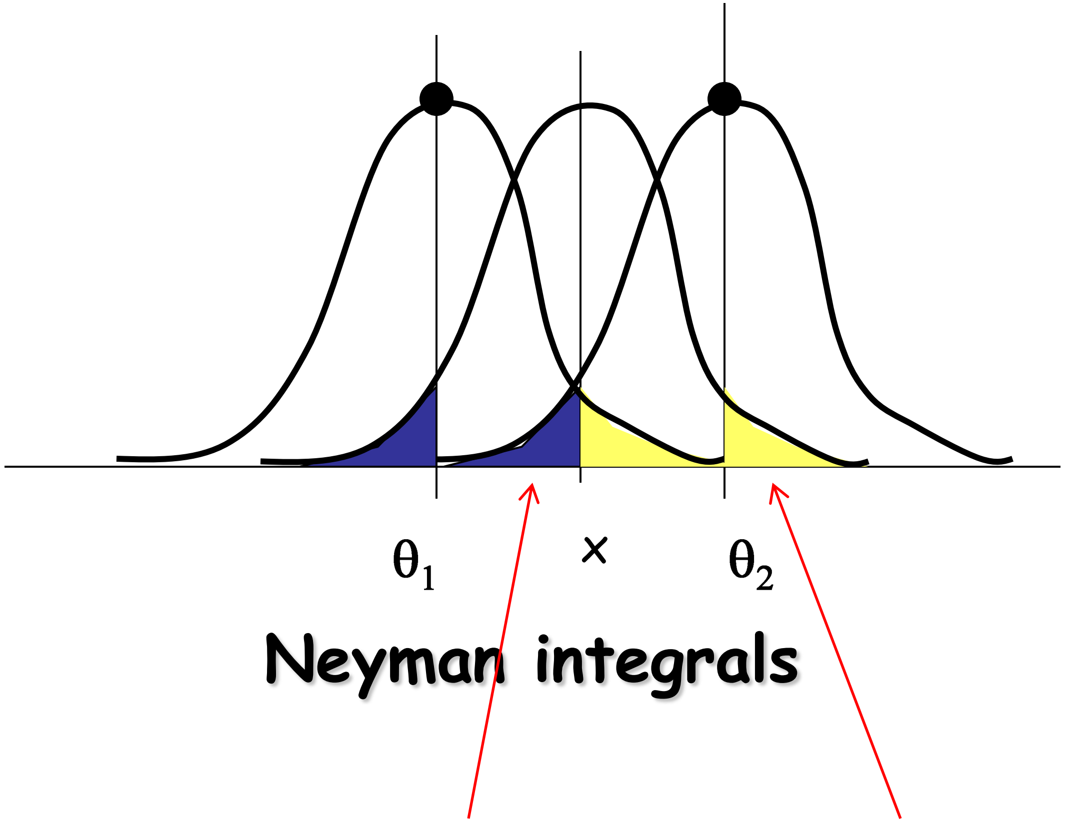
Conclusions

- **Physicists should follow the right statistical notation**

- The usual formulae used by physicists in counting experiments (frequency and efficiency determination) should be abandoned

- the RIGHT formulae for the signal significance there exist and should be used (see **R. Cousins et al, NIM A 595(2008)480**)

END



Neyman integrals

Bootstrap

$$F(x;\theta) = 1 - F(\theta;x)$$

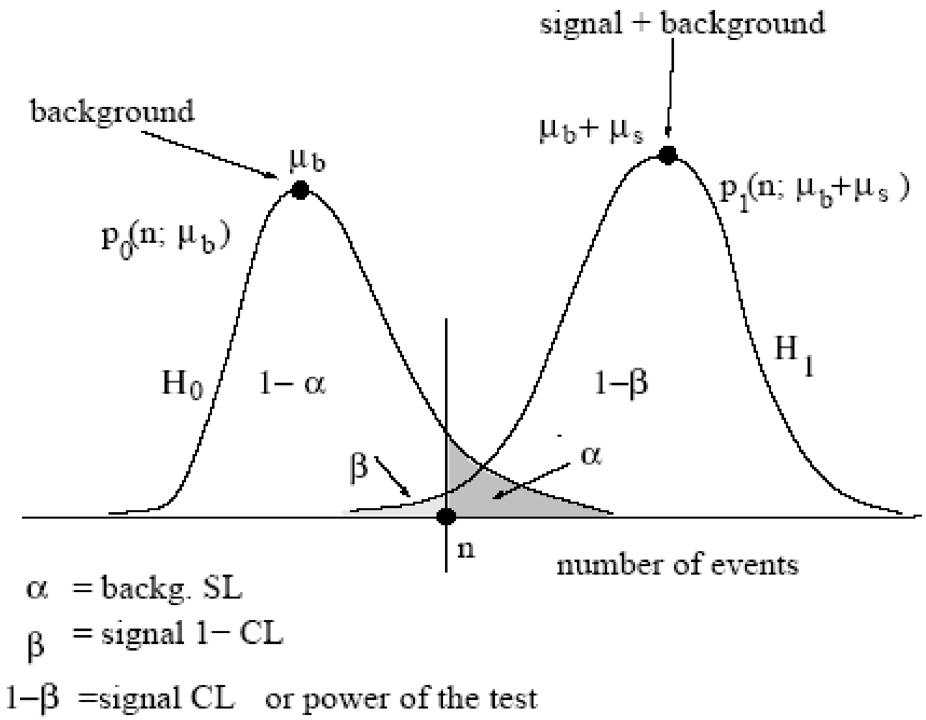
Search for pivotal variables

This method avoids the graphic procedure and the resolution of the Neyman integrals

$\{N = n\}$ **events are observed**, that are supposed to come from a distribution with **expected value** $\mu_b + \mu_s$, where the expected amount of signal μ_s is unknown.

$$p(n, \mu_b) = \frac{\mu_b^n e^{-\mu_b}}{n!} \quad (1)$$

$$p(n, \mu_b + \mu_s) = \frac{(\mu_b + \mu_s)^n}{n!} e^{-\mu_b+\mu_s} \quad (2)$$



From coin tossing to physics: the efficiency measurement

$$P(\varepsilon;k,n) = (n+1)\binom{n}{k}\varepsilon^k(1-\varepsilon)^{n-k}$$
$$= \frac{(n+1)!}{k!\,(n-k)!}\,\varepsilon^k(1-\varepsilon)^{n-k}$$

$$\bar{\varepsilon} = \int_0^1 \varepsilon\, P(\varepsilon;k,n)\; d\varepsilon$$
$$= \frac{(n+1)!}{k!\,(n-k)!}\int_0^1 \varepsilon^{k+1}(1-\varepsilon)^{n-k}d\varepsilon$$
$$= \frac{k+1}{n+2}$$

Valid also for *k=0* and *k=n*

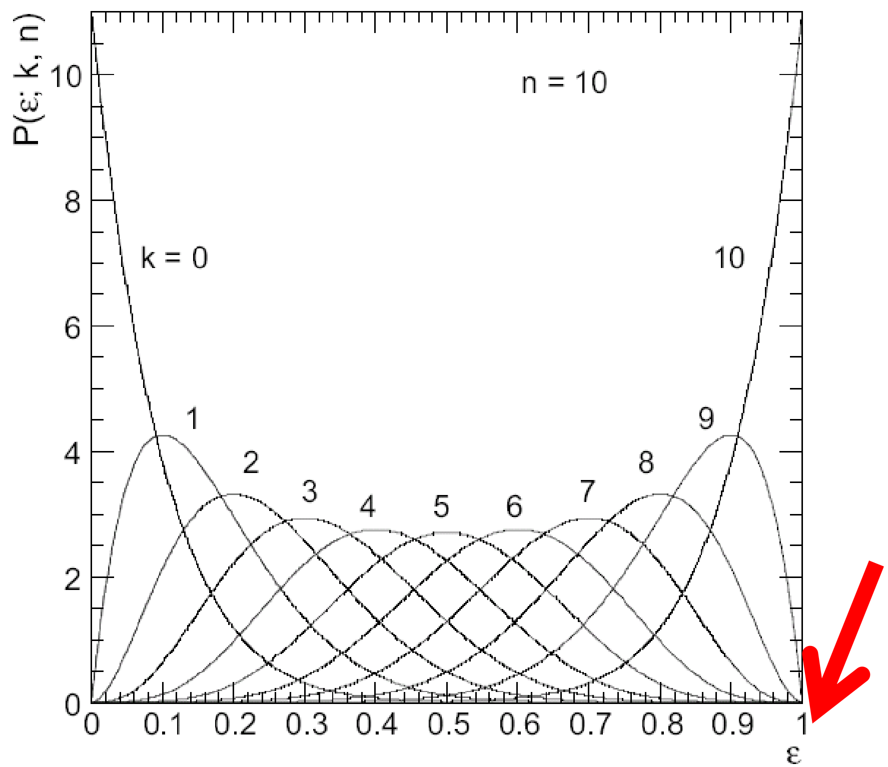
$$V(\varepsilon) = \overline{\varepsilon^2} - \bar{\varepsilon}^2$$
$$= \int_0^1 \varepsilon^2\, P(\varepsilon;k,n)\; d\varepsilon - \bar{\varepsilon}^2$$
$$= \frac{(k+1)\,(k+2)}{(n+2)\,(n+3)} - \frac{(k+1)^2}{(n+2)^2}.$$

ArXiv:physics/0701199v1

Treatment of Errors in Efficiency Calculations

T. Ullrich and Z. Xu
Brookhaven National Laboratory

February 2, 2008



re 1: The probability density function $P(\varepsilon;k,n)$ for $n=10$ and $k=0,1,\ldots,10$.

Frequentist C.I.
right and wrong definitions

RIGHT quotations:

- CL is the probability that the random interval $[T_1, T_2]$ covers the true value θ;
- in an infinite set of repeated identical experiments, a fraction equal to CL will succeed in assigning $\theta \in [\theta_1, \theta_2]$;
- if $\theta \notin [\theta_1, \theta_2]$, one can obtain $\{I = [\theta_1, \theta_2]\}$ in a fraction of experiments $\leq 1 - CL$
- if $H_0 : \theta \notin [\theta_1, \theta_2]$ the probability to reject a true H_0 is $1 - CL$ (falsification).
  see upper and lower limits estimates.

WRONG quotations

- CL is the degree of belief that the true value is in $[\theta_1, \theta_2]$
- $P\{\theta \in [\theta_1, \theta_2]\} = CL$
  (θ is not a random variable!)